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CALCULUS.

410. Proposed by J. A. BULLARD, U. S. Naval Academy.

(*a*) Find the area of the loop of the curve $x^{2q+1} + y^{2q+1} = (2q + 1)ax^q y^q$. (For $q = 1$ we have the folium of Prob. 379.)

(*b*) Find the area between the curve and its asymptote. (From Johnson's *Integral Calculus.*)

SOLUTION BY THE PROPOSER.

The required integration is simplified by the use of polar coördinates. If we let $y = mx$ and express the equation in parametric form we have merely to note that the parameter m is $\tan\theta$ and to substitute in the integral $\frac{1}{2}\int r^2\,d\theta$. Thus in the case of the folium (see p. 343, Vol. XXII) we have

$$A = \frac{1}{2}\int_0^{\pi/2} r^2 d\theta = \frac{1}{2}\int_0^{\infty} x^2 dm = \frac{9a^2}{2}\int_0^{\infty}\frac{m^2 dm}{(1+m^3)^2} = -\frac{3a^2}{2}\left[\frac{1}{(1+m^3)}\right]_0^{\infty} = 3a^2/2.$$

(*a*) The above equation becomes in polar coördinates

$$r = \frac{(2q+1)a\tan^q\theta\sec\theta}{1+\tan^{2q+1}\theta};$$

or in parametric form

$$x = \frac{(2q+1)am^q}{1+m^{2q+1}}, \qquad y = \frac{(2q+1)am^{q+1}}{1+m^{2q+1}},$$

where $m = \tan\theta$. The loop is generated when θ varies from 0 to $\pi/2$. Thus

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/2} r^2 d\theta = \frac{(2q+1)^2a^2}{2}\int_0^{\pi/2}\frac{\tan^{2q}\theta\sec^2\theta d\theta}{(1+\tan^{2q+1}\theta)^2} \\ &= \frac{(2q+1)^2a^2}{2}\int_0^{\infty}\frac{m^{2q}dm}{(1+m^{2q+1})^2} = -\frac{(2q+1)a^2}{2}\left[\frac{1}{1+m^{2q+1}}\right]_0^{\infty} = \frac{(2q+1)a^2}{2}. \end{aligned}$$

(*b*) The equation of the asymptote is $x + y = (-1)^q a$ or

$$r = \frac{(-1)^q a\sec\theta}{1+\tan\theta}.$$

The asymptote forms a triangle of area $a^2/2$ with the coördinate axes, and this triangle separates the remainder of the required area into two equal parts. That part in the second quadrant when q is odd but in the fourth quadrant when q is even, is given by

$$\begin{aligned} A_1 &= \frac{1}{2}\int_{3\pi/4}^{\pi}(r_1^2 - r_2^2)d\theta = \frac{a^2}{2}\int_{-1}^{0}\left[\frac{1}{(1+m)^2} - \frac{(2q+1)^2m^{2q}}{(1+m^{2q+1})^2}\right]dm = \frac{a^2}{2}\left[-\frac{1}{1+m}+\frac{2q+1}{1+m^{2q+1}}\right]_{-1}^{0} \\ &= \frac{a^2}{2}\left[\frac{2q+m-m^2+m^3-\cdots+m^{2q-1}-m^{2q}}{1+m^{2q+1}}\right]_{-1}^{0} \\ &= \frac{a^2}{2}\left[\frac{2q-(2q-1)m+(2q-2)m^2-\cdots+2m^{2q-2}-m^{2q-1}}{1-m+m^2-\cdots-m^{2q-1}+m^{2q}}\right]_{-1}^{0} \\ &= \frac{a^2}{2}\left[2q - \frac{(2q+1)(2q)/2}{2q+1}\right] = \frac{qa^2}{2}. \end{aligned}$$

Then

$$A = 2A_1 + a^2/2 = \frac{(2q+1)a^2}{2}.$$

Thus the area of the loop is equal to the area between the curve and its asymptote.

In the case of the folium ($q = 1$) the coördinate axes trisect the area between the curve and its asymptote.

Also solved by O. S. ADAMS, HORACE OLSON, PAUL CAPRON, and A. M. HARDING.